Hyperelliptic Integrable Systems on
K3 and Rational Surfaces

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Abstract

We show several examples of integrable systems related to special K3 and rational surfaces (e.g., an elliptic K3 surface, a K3 surface given by a double covering of the projective plane, a rational elliptic surface, etc.). The construction, based on Beauvilles's general idea, is considerably simplified by the fact that all examples are described by hyperelliptic curves and Jacobians. This also enables to compare these integrable systems with more classical integrable systems, such as the Neumann system and the periodic Toda chain, which are also associated with rational surfaces. A delicate difference between the cases of K3 and of rational surfaces is pointed out therein.

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1 Introduction

Some ten years ago, Beauville [1] presented a construction of algebraically completely integrable Hamiltonian system (ACIHS) associated with a K3 surface. Beauville's construction is based on Mukai's work [2] on symplectic geometry of moduli spaces of sheaves on K3 surfaces. Beauville considered a special case of Mukai's moduli spaces, and discovered that this moduli space (with a symplectic structure in the sense of Mukai) is an ACIHS, namely, it is a symplectic variety with a Lagrangian fibration whose fibers are Abelian varieties. An interesting byproduct of his result is the fact that this ACIHS has another realization as a symmetric product of the K3 surface.

Beauville's construction suggests a new approach to the issue of construction of integrable systems, namely, an approach from algebraic surfaces. Hurtubise [3] indeed found such a general framework. His result covers many classical integrable systems related to loop algebras [4, 6, 5, 7], Hitchin's integrable systems of Higgs pairs [8], etc., as well as Beauville's integrable systems, on an equal footing. Furthermore, Gorsky et al. [9] pointed out a close relation between this approach and Sklyanin's separation of variables [10].

As for the case of K3 surfaces, however, very few examples of Beauville's integrable systems seem to be *explicitly* known. Almost the only example in the literature is the case of quartics in $\mathbb{P}^3$ that Beauville mentioned in his paper to illustrate his construction. (A more detailed description of this example can be found in the paper of Gorsky et al. [9])

In this paper, we report a few examples of integrable systems related to Beauville's construction. Two of them are direct application of Beauville's construction to an elliptic K3 surface and a K3 surface given by a ramified double covering of $\mathbb{P}^2$. Another example is an integrable system associated with a rational elliptic surface, which is also interesting in comparison with elliptic K3 surfaces. A common characteristic of these examples is that they are described by hyperelliptic curves and Jacobi varieties. This enables us to compare these examples with more classical integrable systems that are related to hyperelliptic curves.

2 Beauville's construction

Following Beauville, we assume that the K3 surface X in consideration has an projective embedding or finite morphism $\phi : X \to \mathbb{P}^g$ by which a g-dimensional family of genus g curves

$C_u = X \cap \phi^{-1}(H_u)$ are cut out by hyperplanes H_u, $u \in (\mathbb{P}^g)^*$. In order to avoid delicate problems, we restrict the hyperplane parameters u to a suitable open subset $\mathcal{U}$ of $(\mathbb{P}^g)^*$. We thus have a family $\mathcal{C} \to \mathcal{U}$ of curves fibered over $\mathcal{U}$.

The phase space of Beauville's integrable system is the associated relative Picard scheme $h : \mathcal{P}ic^g \to \mathcal{U}$ of the degree g component $\mathrm{Pic}^g(C_u)$ of the Picard group of C_u. This relative Picard scheme, upon suitably compactified, can be identified with a moduli space of simple sheaves on X, so that, by Mukai's work, it has a symplectic structure. As Beauville discovered, h is a Lagrangian fibration with respect to Mukai's symplectic structure. Since each fiber $h^{-1}(u) = \mathrm{Pic}^g(C_u)$ is obviously an Abelian variety, the total space $\mathcal{P}ic^g$ of the relative Picard scheme becomes an ACIHS (see the review of Donagi and Markman [11] for the notion of ACIHS).

Another realization of this integrable system is given by the g-fold symmetric product $X^{(g)} = \mathrm{Sym}^g(X)$ equipped with a symplectic structure induced by the complex symplectic structure of X. In order to see the relation with the relative Picard scheme, recall that $\mathrm{Pic}^g(C_u)$ consists of the linear equivalence $[D]$ of an effective divisor $D = p_1 + \cdots + p_g$ of degree g. The unordered g-tuple $(p_1, \ldots, p_g)$ of points then becomes a point of $X^{(g)}$. Conversely, the g-tuple $(p_1, \ldots, p_g)$ of points in a *general position* uniquely determines a hyperplane $H_u \subset \mathbb{P}^g$ (hence a curve $C_u = X \cap H_u$) that passes through the g points. One can thus define the map

$$
\begin{aligned}
X_*^{(g)} &\longrightarrow \mathcal{U} \\
(p_1, \ldots, p_g) &\longmapsto u
\end{aligned}
\tag{1}
$$

from an open subset $X_*^{(g)}$ to $\mathcal{U}$. This map can be lifted up to an open embedding

$$
\begin{aligned}
X_*^{(g)} &\longrightarrow \mathcal{P}ic^g \\
(p_1, \ldots, p_g) &\longmapsto (u, [p_1 + \cdots + p_g])
\end{aligned}
\tag{2}
$$

by adding the linear equivalence class $[p_1 + \cdots + p_g] \in \mathrm{Pic}^g(C_u)$ as the second data. Beauville observed that this is a symplectic mapping. Thus, in particular, $X^{(g)}$ is fibered by the g-fold symmetric product $C_u^{(g)} = \mathrm{Sym}^g(C_u)$ of the curves C_u, and these fibers are Lagrangian.

3 Hyperelliptic integrable system on elliptic K3 surface

We now apply Beauville's construction to an elliptic K3 surface $X \to \mathbb{P}^1$. The Weierstrass model of such an algebraic surface can be written, in affine coordinates, as

$$y^2 = z^3 + f(x)z + g(x), \tag{3}$$

where x is an affine coordinate of $\mathbb{P}^1$ and $f(x)$ and $g(x)$ are polynomials of degree 8 and 12, respectively. The 2-form

$$\omega = \frac{dz \wedge dx}{y}, \tag{4}$$

is nowhere-vanishing and holomorphic, so that X becomes a complex symplectic surface.

3.1 Curves, divisors and symmetric product

We define a five-parameter family of curves C_u, $u = (u_1, \ldots, u_5)$, cut out from X by the equation

$$z = P(x) = \sum_{k=1}^{5} u_k x^{5-k}. \tag{5}$$

In other words, the curves C_u are hyperelliptic curves of genus 5 defined by the equation

$$y^2 = P(x)^3 + f(x)P(x) + g(x). \tag{6}$$

Note that the genus and the number of parameters of curves coincide. Furthermore, these curves are hyperplane sections of the following finite morphism $\phi : X \to \mathbb{P}^5$:

$$
\begin{aligned}
X &\longrightarrow \mathbb{P}^5 \\
(x, y, z) &\longmapsto (-z : x^4 : x^3 : x^2 : x^1 : x : 1)
\end{aligned} \tag{7}
$$

Although we have used the affine coordinates for simplicity, it is not difficult to projectivize this definition. The five parameters $u_1, \ldots, u_5$ thus turn out to be affine coordinates of $\mathbb{P}^5$.

The mapping $X_*^{(5)} \to \mathcal{U}$ takes a particularly simple form in the present setting. Given an unordered 5-tuple $(p_1, \ldots, p_5)$ of points of X, let (x_j, y_j, z_j) denote the affine coordinates of the j-th point p_j. The curve C_u passes through the five points if and only if the equations

$$P(x_j) = z_j \tag{8}$$

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are satisfied for $j = 1, \ldots, 5$. The polynomial $P(x)$ is uniquely determined by these conditions if the x_j's are distinct: $x_j \neq x_k$ ($j \neq k$). In fact, the Lagrange interpolation formula provides an explicit expression of $P(x)$:

$$P(x) = \sum_{j=1}^{5} z_j \prod_{k \neq j} \frac{x - x_k}{x_j - x_k}. \tag{9}$$

This determines u_j's as rational functions of $(x_1, \ldots, x_5, z_1, \ldots, z_5)$, e.g.,

$$u_1 = \sum_{j=1}^{5} z_j / \prod_{k \neq j} (x_j - x_k). \tag{10}$$

These functions are to be a commuting set of Hamiltonians.

3.2   Abel-Jacobi mapping and symplectic form

The symplectic structure of $X^{(5)}$ is defined by the 2-form

$$\Omega = \sum_{j=1}^{5} \frac{dz_j \wedge dx_j}{y_j}, \tag{11}$$

where (x_j, y_j, z_j) denote the affine coordinates of the p_j's. According to Beauville, this symplectic structure is mapped to Mukai's symplectic structure on $\mathcal{P}ic^5$.

Following Hurtubise, we now slightly modify Beauville's construction: We shift the divisors $p_1 + \cdots + p_5$ to $p_1 + \cdots + p_5 - 5p_\infty$, where p_∞ is the point $(x, y, z) = (\infty, \infty, \infty)$ of C_u, and consider the relative Jacobian $\mathcal{J}ac \to \mathcal{U}$ of the Jacobi varieties $\mathrm{Jac}(C_u)$ fibered over $\mathcal{U}$ rather than the relative Picard. The Abel-Jacobi mapping on C_u then induces a mapping $X_*^{(5)} \to \mathcal{J}ac$, which plays the role of Beauville's mapping.

The Abel-Jacobi mapping sends the symmetric product $C_u^{(5)}$ onto the Jacobi variety $\mathrm{Jac}(C_u) = \mathbb{C}^5 / L_u$ realized as a complex torus with period lattice L_u:

$$\begin{array}{ccc} C_u^{(5)} & \longrightarrow & \mathrm{Jac}(C_u) = \mathbb{C}^5 / L_u \\ (p_1, \ldots, p_5) & \longmapsto & (\psi_1, \ldots, \psi_5) \end{array} \tag{12}$$

In order to define this mapping, one has to choose a basis of holomorphic differentials on C_u. We take the standard holomorphic differentials

$$\sigma_k = \frac{x^{5-k} dx}{y} \quad (k = 1, \ldots, 5). \tag{13}$$

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The ψ_k's are then given by a sum of integrals of the σ_k's:

$$\psi_k = \sum_{j=1}^{5} \int_{(\infty,\infty)}^{(x_j,y_j)} \frac{x^{5-k} dx}{y} \tag{14}$$

Now $(u_1, \dots, u_5, \psi_1, \dots, \psi_5)$ give a local coordinate system on the relative Jacobian $\mathcal{J}ac$. Remarkably, Ω turns out to take a canonical form in these coordinates:

$$\Omega = \sum_{k=1}^{5} du_k \wedge d\psi_k. \tag{15}$$

In other words, the "toroidal" coordinates ψ_k' are nothing but the canonical conjugate variables of the hyperplane parameters u_k.

3.3 Proof of (15)

Since

$$z_j = P(x_j) = \sum_{k=1}^{5} u_k x_j^{5-k},$$

Ω can be written

$$\Omega = \sum_{j,k=1}^{5} \frac{du_k \wedge x_j^{5-k} dx_j}{y_j}. \tag{16}$$

It is convenient choose $(x_1, \dots, x_5, \psi_1, \dots, \psi_5)$, rather than $(u_1, \dots, u_5, \psi_1, \dots, \psi_5)$, as independent variables here. The total differential of $d\psi_k$ can be now expressed as

$$d\psi_k = \sum_{j=1}^{5} \frac{\partial \psi_k}{\partial x_j} dx_j + \sum_{\ell=1}^{5} \frac{\partial \psi_k}{\partial u_\ell} du_\ell,$$

so that

$$\sum_{k=1}^{5} du_k \wedge d\psi_k = \sum_{j,k=1}^{5} \frac{\partial \psi_k}{\partial x_j} du_k \wedge dx_j + \sum_{k,\ell=1}^{5} \frac{\partial \psi_k}{\partial u_\ell} du_k \wedge du_\ell. \tag{17}$$

Differentiating the explicit definition

$$\psi_k = \sum_{j=1}^{5} \int_{\infty}^{x_j} \frac{x^{5-k} dx}{\left(P(x)^3 + f(x)P(x) + g(x)\right)^{1/2}}$$

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of the Abel-Jacobi mapping gives

$$\frac{\partial \psi_k}{\partial x_j} = \frac{x_j^{5-k}}{y_j}, \tag{18}$$

$$\frac{\partial \psi_k}{\partial u_\ell} = \sum_{j=1}^{5} \int_{\infty}^{x_j} \frac{x^{5-k} \cdot x^{5-\ell}\Big(3P(x)^2 + f(x)\Big)dx}{2\Big(P(x)^3 + f(x)P(x) + g(x)\Big)^{3/2}}. \tag{19}$$

In particular,

$$\frac{\partial \psi_k}{\partial u_\ell} = \frac{\partial \psi_\ell}{\partial u_k}, \tag{20}$$

therefore

$$\sum_{k,\ell=1}^{5} \frac{\partial \psi_k}{\partial u_\ell} du_k \wedge du_\ell = 0. \tag{21}$$

This implies the equality

$$\sum_{k=1}^{5} du_k \wedge d\psi_k = \sum_{j,k=1}^{5} \frac{x_j^{5-k}}{y_j} du_k \wedge dx_j = \Omega, \tag{22}$$

thus completing the proof of (15).

3.4   Remarks

1.   Essentially the same expression of the symplectic form is stated in Hurtubise's paper [3] in a more general form. This expression is thus not specific to the case of hyperelliptic curves, but rather universal.

2.   (15) is a concise expression of the fact that the relative Jacobian $\mathcal{J}ac \to \mathcal{U}$ is an ACIHS. Firstly, the involutivity (Poisson-commutativity) of the "Hamiltonians" u_k is obvious from this expression of Ω. Accordingly, the intersection of the level surfaces of the u_k's, which are nothing but the Jacobi varieties $\mathrm{Jac}(C_u)$, are Lagrangian subvarieties. Lastly, these Hamiltonians generate a commuting set of *linear* flows on these complex tori. The Hamiltonian flow of u_j is indeed given by

$$\psi_k(t) = \psi_k(0) + \delta_{jk}t. \tag{23}$$

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3. Ω should be invariant under the shift

$$\psi_k \longrightarrow \psi_k + e_k \tag{24}$$

by any element $e = (e_1, \ldots, e_5)$ of the period lattice L_u. This implies the nontrivial equations

$$\sum_{k=1}^{5} du_k \wedge de_k = 0 \tag{25}$$

among the components of the vector e (which depends on u). This is an avatar of the "cubic condition" of Donagi and Markman [11]. Actually, we can verify these conditions directly: Note that e_k is given by the period integral

$$e_k = \int_\gamma \frac{x^{5-k} dx}{y} \tag{26}$$

along a cycle γ. From this expression, one can derive the equality

$$\frac{\partial e_k}{\partial u_\ell} = \frac{\partial e_\ell}{\partial u_k} \tag{27}$$

in the same way as the calculations in the proof of (15).

4. A one-parameter family of hyperelliptic curves of genus 5 on the elliptic K3 surface appears in the work of Fock et al. on duality of integrable systems [12]. In fact, their hyperelliptic curves are the subfamily of ours with $u_1 = u_2 = u_3 = u_4 = 0$.

4 Hyperelliptic integrable system on non-elliptic K3 surface

We here present an example associated with a double covering of $\mathbb{P}^2$. Let (x, z) be affine coordinates of $\mathbb{P}^2$ and consider an algebraic surface X defined by the equation

$$y^2 = f(x, z), \tag{28}$$

where $f(x, z)$ is a polynomial of degree six whose zero-locus $f(x, z) = 0$ is a nonsingular sextic curve on $\mathbb{P}^2$. This is a K3 surface that appears in Mukai's work as an interesting example [2]. A nowhere-vanishing holomorphic 2-form is given by

$$\omega = \frac{dz \wedge dx}{y}. \tag{29}$$

A two-parameter family of curves C_u, $u = (u_1, u_2)$, can be cut out from X by the equation

$$z = P(x) = u_1 x + u_2. \tag{30}$$

In other words, C_u is the hyperelliptic curve of genus 2 defined by the equation

$$y^2 = f(x, u_1 x + u_2). \tag{31}$$

Note that these curves are nothing but the hyperplane sections associated with the double covering $\phi : X \to \mathbb{P}^2$.

The rest of the construction is fully parallel to the case of the elliptic K3 surface. The integrable system is realized on (an open subset of) the 2-fold symmetric product $X^{(2)}$ or on the relative Jacobian $\mathcal{J}ac$. Let (p_1, p_2) denote an unordered pair of points of X, (x_j, y_j, z_j) the coordinates of p_j, and Ω the symplectic form induced on $X^{(2)}$:

$$\Omega = \sum_{j=1,2} \frac{dz_j \wedge dx_j}{y_j}. \tag{32}$$

The Abel-Jacobi mapping

$$\begin{aligned} C_u^{(2)} &\longrightarrow \operatorname{Jac}(C_u) = \mathbb{C}^2 / L_u \\ (p_1, p_2) &\longmapsto (\psi_1, \psi_2) \end{aligned} \tag{33}$$

is now given by

$$\psi_1 = \sum_{j=1,2} \int_{(\infty,\infty)}^{(x_j,y_j)} \frac{x\,dx}{y}, \quad \psi_2 = \sum_{j=1,2} \int_{(\infty,\infty)}^{(x_j,y_j)} \frac{dx}{y}. \tag{34}$$

The symplectic form Ω takes the canonical form

$$\Omega = \sum_{k=1,2} du_k \wedge d\psi_k \tag{35}$$

in the coordinates $(u_1, u_2, \psi_1, \psi_2)$.

5 Hyperelliptic integrable system on rational surface

5.1 Integrable system associated with rational elliptic surface

If the polynomials $f(x)$ and $g(x)$ in the Weierstrass model of elliptic K3 surfaces are replaced by polynomials of degree 4 and 6, resplectively, the outcome is an rational elliptic surface.

The 2-form

$$\omega = \frac{dz \wedge dx}{y} \tag{36}$$

is nowhere-vanishing and holomorphic in the *affine part* of the surface, having poles at the compactification divisor at infinity. Let us examine this case in comparison with the case of K3 surfaces.

Simple power counting suggests that the most natural choice of $P(x)$ in this case will be a polynomial of degree 2:

$$P(x) = u_0 x^2 + u_1 x + u_2. \tag{37}$$

The equation $z = P(x)$ then defines a hyperelliptic curve of genus 2 in X:

$$y^2 = P(x)^3 + f(x)P(x) + g(x). \tag{38}$$

Ganor [13] considered the same family of curves in the context of string theory. The foregoing construction, however, does not work literally, firstly because the number of parameters (three) and the genus of the curves (two) do not match.

A correct prescription is to fix u_0 to a constant c, and to take the two-parameter family of curves C_u, $u = (u_1, u_2)$, cut out from X by the equation

$$z = P(x) = cx^2 + u_1 x + u_2. \tag{39}$$

In other words, $u_0 = c$ should be treated as a *Casimir function* on a Poisson manifold.

Now we consider $X^{(2)}$ (rather than $X^{(3)}$ that Ganor argued) with the symplectic form

$$\Omega = \sum_{k=1,2} \frac{dz_j \wedge dx_j}{y_j}, \tag{40}$$

where (x_j, y_j, z_j) are the coordinates of an unordered pair (p_1, p_2) of points on X. The Abel-Jacobi mapping $(p_1, p_2) \mapsto (\psi_1, \psi_2)$ is given by

$$\psi_1 = \sum_{j=1,2} \int_{(\infty,\infty)}^{(x_j,y_j)} \frac{x dx}{y}, \quad \psi_2 = \sum_{j=1,2} \int_{(\infty,\infty)}^{(x_j,y_j)} \frac{dx}{y}, \tag{41}$$

and sends Ω the canonical form:

$$\Omega = \sum_{k=1,2} du_k \wedge d\psi_k. \tag{42}$$

Thus the situation for the rational elliptic surfaces is slightly different from the case of K3 surfaces. Namely, we have to fix the leading coefficient of $P(x)$ to a constant c so as to match the number of parameters and the genus of curves.

5.2 Neumann system and rational surface

It is instructive to compare the foregoing construction with the Neumann system. The Neumann system is a classical integrable system that was solved in the 19th century [14] using the theory of hyperelliptic integrals developed in those days, and revived by Moser [15] from a modern point of view.

The hyperelliptic curves of the Neumann system take the special form

$$y^2 = P(x)Q(x), \tag{43}$$

where $P(x)$ and $Q(x)$ are polynomials of degree N and $N+1$ of the form

$$P(x) = x^N + \sum_{k=1}^{N} u_k x^{N-k}, \quad Q(x) = \prod_{n=1}^{N+1} (x - c_n), \tag{44}$$

and N, the genus of the curves, is equal to the degrees of freedom of the Neumann system. The coefficients u_k of $P(x)$ are integrals of motion in involution. The c_n's are non-dynamical structural constants (i.e., Casimir functions) of the system.

One will soon notice that a rational surface is hidden behind these hyperelliptic curves. This rational surface X is defined by the equation

$$y^2 = zQ(x). \tag{45}$$

The 2-form

$$\omega = \frac{dz \wedge dx}{y} \tag{46}$$

is nowhere-vanishing and holomorphic in the affine part of X. The curves C_u cut out from X by the equation $z = P(x)$ are exactly the aforementioned hyperelliptic curves. It is not difficult to confirm that the construction based on the symmetric product $X^{(N)}$ or the relative Jacobian $\mathcal{J}ac$ does reproduce the classical method for solving the Neumann system.

It is interesting to note that the leading coefficient of $P(x)$, which is now set to 1, is also a kind of structural constant. This constant is related to the radius of an N-dimensional sphere on which the Neumann system is realized as a mechanical system. If the radius takes a different value, the leading coefficient of $P(x)$ accordingly varies as

$$P(x) = cx^N + \sum_{k=1}^{N} u_k x^{N-k}.$$

This is the same situation that we have encountered in the case of rational elliptic surfaces.

6   Concluding Remarks

We have shown several integrable systems that illustrate Beauville's construction. The construction is considerably simplified by the fact that the curves arising therein are hyperelliptic. Nevertheless, the most essential part, i.e., the two expressions of Ω on the symmetric product $X^{(g)}$ and on the relative Jacobian $\mathcal{J}ac$, are rather universal and can be extended to a non-hyperelliptic case. Let us stress that the coordinates (x_j, y_j, z_j) of the unordered g-tuple $(p_1, \dots, p_g) \in X^{(g)}$ correspond to "separated variables" in Sklyanin's separation of variables.

We have pointed out a delicate difference between the cases of K3 and of rational surfaces. Namely, some of the hyperplane parameters u_k in the latter case have to be interpreted as a Casimir rather than an integral of motion. This issue deserves to be pursued further.

The present approach from algebraic surfaces can be a promising alternative to the conventional method based on a Lax representation. We have seen such a possibility in the case of the Neumann system. Let us mention another example, which is related to Witten's eleven-dimensional interpretation [16] of the Seiberg-Witten curves for four-dimensional $N = 2$ supersymmetric QCD.

The Seiberg-Witten curves for $\mathrm{SU}(N_c)$ gauge theory coupled to N_f ($N_f \leq 2N_c$) hypermultiplets are hyperelliptic curves of genus $g = N_c - 1$ defined by the equation

$$y^2 - P(x)y + Q(x) = 0. \tag{47}$$

Here $P(x)$ is a polynomial of the form

$$P(x) = x^{N_c} + \sum_{k=2}^{N_c} u_k x^{N_c - k}, \tag{48}$$

and $Q(x)$ a polynomial of degree N_f of the form

$$Q(x) = \Lambda^{2N_c - N_f} \prod_{\ell=1}^{N_f} (x + m_\ell). \tag{49}$$

The u_k's are the "moduli" of the Seiberg-Witten curves, which we interpret as (part of) the parameters of hyperplane sections. If $N_f = 0$, $Q(x)$ reduces to a constant, and the curve is exactly the the spectral curve of the N_c-periodic Toda chain.

In Witten's interpretation, these curves are embedded into the *affine* rational surface X defined by the equation

$$yz = Q(x), \tag{50}$$

A nowhere-vanishing holomorphic 2-form on X is given by

$$\omega = \frac{dy \wedge dx}{y} = \frac{dx \wedge dz}{z}. \tag{51}$$

In fact, this complex surface is a realization of the multi-Taub-NUT space (gravitational instanton) in one of its 2-parameter family of complex structures [17]. As noted by Nakatsu et al. [18] and de Boer et al. [19], the Seiberg-Witten curves are cut out from X by the equation

$$y + z = P(x). \tag{52}$$

One can now repeat the foregoing construction with a slightest modification to obtain an integrable system realized on $X^{(g)}$ or on $\mathcal{J}ac$. Presumably, this integrable system will be equivalent to the spin chain that Gorsky et al. [20] proposed as an integrable system for these Seiberg-Witten curves. Our construction, however, is more direct and seemingly more natural in the context of brane theory [9, 16].

Acknowledgements

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